Metric criteria for fixed price of countable groups

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Main Result

$\boxed{\text{THM}}$ (B, Bowen)

$\boxed{\begin{array}{l}\text{If } \Gamma \text{ is an infinite countable group with an infinite measure preserving (imp) action which is limit-amenable,} \\ \text{partially doubly recurrent, and has normalized cost } p, \text{ then } \Gamma \text{ has max cost at most } p.\end{array}}$

$\Rightarrow$ if $p = 1$, then Γ has fixed price 1.

Application

$\boxed{\text{Thm}}$ Let Γ_1, Γ_2 be finitely generated groups such that the word metrics d_1, d_2 have "roughly comparable growth" rates. Then $\Gamma = \Gamma_1 \times \Gamma_2$ has fixed price 1.

$\boxed{\text{Corollary}}$ $\Gamma \times \Gamma$ has fixed price 1.

$\boxed{\text{Remark}}$

$\boxed{\text{Thm}}$ (Knezeli, 2025)

$\boxed{\text{Let } \Gamma_1, \Gamma_2 \text{ be infinite countable groups. Then } \Gamma = \Gamma_1 \times \Gamma_2 \text{ has fixed price } 1.}$

- proven independently, uses some similar ideas
- we work to build a broader framework

Our Inspiration

THM (Fraczyk-Mellick-Wilkens '23)
Higher rank semisimple Lie groups and products of automorphism groups of trees have fixed price 1.

Proof sketch

These groups exhibit a certain "boundary" action (corona action) which is amenable and doubly recurrent. Apply:

> **THM** (F-M-W)
> If G is a unimodular, lcsc group with amenable and doubly recurrent corona action, then G has fixed price 1.

Generalized actions

THM (Mellick '23)
Suppose $A < G$ is an amenable, closed, unimodular, noncompact subgroup such that $G \curvearrowright G/A$ is doubly recurrent. Then G has fixed price one.

→ relies on subgroup structure, rather than Lie group structure

→ recovers some of FMW

How do these results compare to our own?

generally: move concepts from lcsc case to discrete case

- don't use specific Lie group or subgroup structure

- generalize amenability to limit-amenability and double recurrence to partial double recurrence

- produce actions satisfying thm for some, but not all, product groups

$\boxed{\text{Cost}}$

Let $R \subset X \times X$ be a pmp countable Borel equivalence relation

Let G_R be a graphing generating the equiv rln. R, i.e. the connected components of G_R are the equivalence classes of R.

Then the cost of G_R is

$$\operatorname{Cost}(G_R) = \frac{1}{2} \int \deg_{G_R}(x)\, \mu(x).$$

$\boxed{\text{Cost}}$ (cont.)

The cost of an equivalence relation R is
$\inf\{\text{cost}(G) : G \text{ is a graphing generating } R\}$

The cost of a countable group Γ is
$\inf\{\text{cost}(R) : R \text{ is induced by a pmp free Borel action of } \Gamma\}$

idea: cost is a measurable analogue to free rank

Cost

Examples

- The cost of any infinite amenable group is 1.
  This is due to the hyperfiniteness.

- The cost of $F_n = n$ for the free group on n generators.
  This is equivalent to the rank of F_n.

$\boxed{\text{Cost}}$ (cont.)

The max-cost of a group Γ is the maximal cost among all its ~~ess~~ ess. free, pmp actions.

A group is said to have fixed price if its cost and max-cost agree.

Weak containment

$\alpha : \Gamma \curvearrowright (X, \mu)$, $\beta : \Gamma \curvearrowright (Y, \nu)$ pmp actions

Idea: α is weakly contained in β, denoted $\alpha \prec \beta$, if the action of α on finitely many group elements and finitely many Borel sets can be approx. by β

Weak containment

$\alpha : \Gamma \curvearrowright (X, \mu)$, $\beta : \Gamma \curvearrowright (Y, \nu)$ pmp actions

We say α is weakly contained in β, denoted $\alpha \prec \beta$, if every finite coloring $\phi : X \to A$, finite $F \subset \Gamma$, and $\varepsilon > 0$, $\exists$ a coloring $\psi : Y \to A$ such that

$$\sum_{a \in A} \sum_{b \in A} \sum_{f \in F} \Big| \mu(\{x \in X : \phi(x) = a,\ \phi(f x) = b\}) - \nu(\{y \in Y : \psi(y) = a,\ \psi(f y) = b\}) \Big| < \varepsilon .$$

Weakly Bernoulli actions

An action a is weakly Bernoulli if it is weakly contained in a Bernoulli action.

$\boxed{\text{Thm}}$ (Abert-Weiss, 2013)
If a pmp ess. free action is weakly Bernoulli, then its cost is the max-cost.

Sketch

1. All essentially free pmp actions weakly contain all Bernoulli shifts
2. Cost is monotonic under weak containment (Kechris)

Infinite measure preserving actions

An imp (infinite measure preserving) action is an action of Γ on a std σ-finite infinite measure space (X, μ) by measurable automorphisms

$\boxed{\text{Ex}}$ Let $H \leq \Gamma$ have infinite index, equip Γ/H with Haar measure. Then $\Gamma \curvearrowright \Gamma/H$ is an imp action.

POV: relationship $\Gamma \curvearrowright \Gamma/H \longleftrightarrow$ ergodic actions

H finite index $\Rightarrow$ pmp $\qquad$ H infinite index $\Rightarrow$ imp

Weak containment for imp actions

Let A finite, $A_* = A \cup \{*\}$ where $* \in A$. Then
$\phi : X \to A_*$ is (μ, A)-finite if $\mu(\{x \in X : \phi(x) \neq *\}) < \infty$.

Idea:
$\alpha \prec \beta$ if α can be approx by β on all "finite windows".

$\boxed{\text{Defn}}$ A mp action α is weakly contained in β
if $\forall$ finite A, (μ, A)-finite measurable map $\phi : X \to A^*$,
finite $F \subset \Gamma$, and $\varepsilon > 0$, $\exists$ a (ν, A)-finite measurable
map $\psi : Y \to A_*$ s.t.

$$\sum_{a \in A} \sum_{b \in A^*} \sum_{f \in F} \Big| \mu(\{x \in X : \phi(x) = a,\ \phi(fx) = b\}) - \nu(\{y \in Y : \psi(y) = a,\ \psi(fy) = b\}) \Big| < \varepsilon.$$

Amenable actions

Def An imp action $\Gamma \curvearrowright (X, \mu)$ is (Zimmer)-amenable R_Γ is hyperfinite mod μ AND

$\mathrm{Stab}_\Gamma(x) = \{g \in \Gamma : gx = x\}$ is amenable for a.e. x.

↳ stronger statement than amenability for equiv. relations

Examples

- All actions of amenable groups are amenable.
- Let $A < \Gamma$ be amenable. Then $\Gamma \curvearrowright \Gamma / A$ is amenable.
- The boundary action $SL(2, \mathbb{Z}) \curvearrowright \partial \mathbb{H}^2 \cong S^1$ is amenable.

Limit-amenability

An imp action is *limit-amenable* if it is the vague limit of mp factors of amenable actions,

i.e. $\exists\, (\mu_n)_{n=1}^{\infty}$ with $\mu_n \longrightarrow \mu$ vaguely where $\Gamma \curvearrowright (X, \mu_n)$ is a mp factor of an amenable action.

THM

$\boxed{\text{Limit-amenable imp actions are weakly contained in the class of amenable imp actions.}}$

Limit-regularity

An imp action is *regular* if it is measurably conjugate to the left action $\Gamma \curvearrowright (\Gamma, \lambda_{Haar})$.

It is *limit-regular* if it is the vague limit of mp factors of regular actions.

THM An imp action is *limit-amenable* $\Longleftrightarrow$ it is *limit-regular*.

RMKs

① If $\Gamma \curvearrowright (X, \mu)$ is limit-amenable and $\mu(X) < \infty \Rightarrow \Gamma$ is amenable.

② If Γ is exact, then limit-amenable $\Rightarrow$ amenable.

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Point Processes

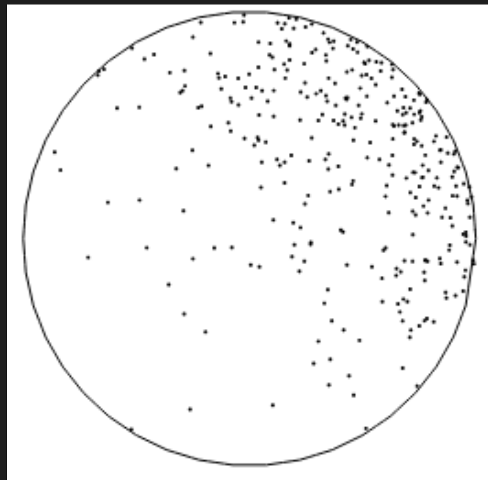
A point measure on lcsc space X is a locally-finite sum of Dirac measures $\sum c(x)\,\delta_x$ with $c(x) \in \mathbb{Z}_{\geq 0}$.

Denote by $\mathbb{M}(X)$ the set of all point measures on X.

A point process on X is an $\mathbb{M}(X)$-valued random variable

$$\Pi : (X, \mu) \longrightarrow \mathbb{M}(X).$$

Its law is $\Pi_*(\mu)$.



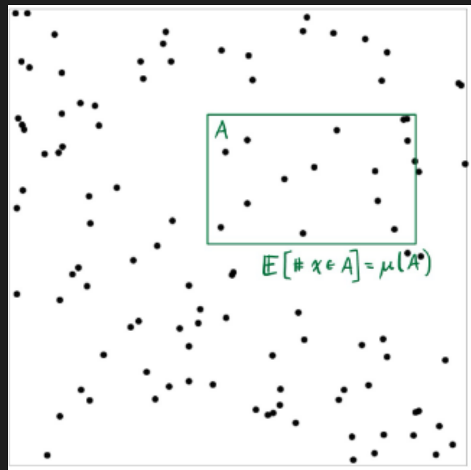
Realization of a non-homogenous point process.

Poisson point processes

A poisson point process on X with intensity μ is a point process Π s.t.:

1. For any msbl $E \subset X$ with $\mu(X) < \infty$, $\Pi(E)$ is a Poisson random variable with mean $\mu(E)$

2. $E_1, E_2, \ldots \subseteq X$ pointwise disjont, then the restrictions $\Pi|_{E_i}$ are jointly independent r.v.



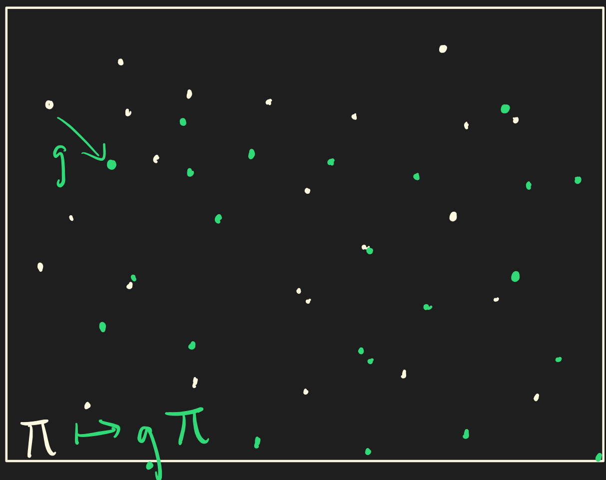
Poisson point process on $\mathbb{R}^2$.

All such processes have the same law, denoted $\mathrm{Pois}(\mu) \in \mathrm{Prob}(\mathbb{M}(X))$.

Poisson suspensions

Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be imp. The Poisson suspension of α is the pmp induced action $\mathrm{Pois}(\alpha) : \Gamma \curvearrowright (\mathbb{M}(X), \mathrm{Pois}(\mu))$

$$\pi \longmapsto g \cdot \pi, \quad x \longmapsto gx \quad \forall\, x \in \pi$$



action $\mathrm{Pois}(\alpha)$

for $g \in \Gamma$, $\pi \in \mathbb{M}(X)$

Poisson suspensions of limit-amenable imp actions

THM

If α a limit-amenable imp action, then $\mathrm{Pois}(\alpha)$ is weakly contained in Bernoulli.

Sketch of proof.

① Show $\mathrm{Pois} : \mathrm{Radon}(X) \to \mathrm{Prob}(\mathbb{M}(X))$ is weak*-cont.

② Show α is regular $\Rightarrow$ $\mathrm{Pois}(\alpha)$ iso. to a Bernoulli shift

$\Rightarrow$ α limit-regular $\Rightarrow$ $\mathrm{Pois}(\alpha)$ weakly Bernoulli. □

$\Rightarrow$ By Abert-Weiss, $\mathrm{Pois}(\alpha)$ has max cost.

Normalized Cost

$\Gamma \curvearrowright (X, \mu)$ ess. free imp action, R the orbit-equiv rln.

Define the normalized cost to be

$$\mathrm{ncost}(\Gamma, X, \mu) = \mathrm{cost}_\mu(R_S) + 1 - \mu(S)$$

where $S \subset X$ is a complete section for R with $0 < \mu(S) < \infty$.

NOTE: This is well defined. Let S_1, S_2 be two complete sections. Then $\mathrm{Cost}(R_{S_1}) - \mu(S_1) = \mathrm{Cost}(R_{S_2}) - \mu(S_2)$ by a result of Gaboriau.

Double Recurrence

Given $Y \subset X$, $x \in X$, let $\mathrm{Ret}(Y, x) = \{ g \in \Gamma : gx \in Y \}$.

- Y is recurrent if for a.e. $y \in Y$, $\mathrm{Ret}(Y, y) = \infty$
- $\Gamma \curvearrowright (X, \mu)$ is conservative if $\forall\, Y \subset X$ msbl, Y is recurrent
- $\Gamma \curvearrowright (X, \mu)$ is doubly reurrent if the diagonal action $\Gamma \curvearrowright (X^2, \mu^2)$ is conservative

Partial Double Recurrence

$\Gamma \curvearrowright (X, \mu)$ is partially doubly recurrent (PDR) if for a.e. pair $x, y \in X$ there exists a chain of elements $x = x_0, x_1, \ldots, x_n = y$ such that (x_i, x_{i+1}) lies in an infinitely recurrent component of $X \times X$ for the diagonal action $\Gamma \curvearrowright (X^2, \mu^2)$.

Cost and Poisson suspensions

Let R_ρ be the equiv rln induced by $\Gamma \curvearrowright (X, \mu)$
R_π ———————— " ———————— by $\Gamma \curvearrowright (\mathbb{M}(X), \mathrm{Pois}(\mu))$.

THM Let $\Gamma \curvearrowright (X,\mu)$ be an imp action such that a.e. ergodic component is infinite, ess. free, non-atomic, and PDR. Then

$$\mathrm{cost}_{\mathrm{Pois}(\mu)}(R_\pi) \leq \mathrm{ncost}(R_\rho).$$

Proof Sketch

⓪ WLOG, assume the action is ergodic.

① Show $\Gamma \curvearrowright (X,\mu)$ ergodic & ess. free $\Rightarrow \Gamma \curvearrowright (\mathbb{M}(X), \mathrm{Pois}(\mu))$ ess. free.

② Choose a finite measure complete section $S \subset X$, and a graphing $\mathcal{G}_S \subset R_{\Gamma|S}$ with $\mathrm{Cost}_{\mu|S}(\mathcal{G}) < \mathrm{cost}(R_\Gamma|S) + \varepsilon$

③ For $x \in X$, create connected & Γ-equivariant graphs in $\Gamma \times \Gamma$

$V(x) = \mathrm{Ret}(S, x)$, $E(x) = \{(f,g) : (f^{-1}x, g^{-1}x) \in \mathcal{G}\}$

idea: Represent $\mathcal{G}_S$ by a subset in $\mathrm{Graph}(\Gamma)$: the set of symmetric subsets of $\Gamma \times \Gamma$

④ Lift to $\mathbb{M}(X) \subset \mathrm{Graph}(\Gamma)$ by letting

$V(\pi) = \bigcup \{V(x) : x \in X,\ \pi(x) > 0\}$, $\hat{E}(\pi) = \bigcup \{E(x) : x \in X,\ \pi(x) > 0\}$

Proof Sketch

5. Connect with a small Bernoulli edge percolation $B_p \subset \mathrm{Graph}(\Gamma)$ on $\Gamma \times \Gamma$, where for each $g, h \in \Gamma$ the edge $\{g, gh\}$ is present with probability $p(h)$. Denote the law as $\nu_p \in \mathrm{Prob}(\mathrm{Graph}(\Gamma))$.

6. Use PDR to show that $\hat{E}(\pi)$ is contained in a single connected component of $\hat{E}(\pi) \cup B_p$ w.p. 1.

7. Cost of $\Gamma \curvearrowright (\mathbb{M}(X), \mathrm{Pois}(\mu))$ is equal to the cost of $\Gamma \curvearrowright (\mathbb{M}(X) \times \mathrm{Graph}(\Gamma), \mathrm{Pois}(\mu) \times \nu_p)$ b/c they're weakly equiv.

8. Bound the cost. Computations give

$$\mathrm{cost}(R_\pi) \leq n\mathrm{cost}(R_p) + e^{-\mu(S)} - 1 + \mu(S).$$

Let $\mu(S) \searrow 0$.

Proof of Main Theorem

THM If Γ has a limit-amenable, PDR imp action with normalized cost p, then Γ has max cost $\leq p$.

Proof ① Show a.e. ergodic component limit-amenable & PDR
→ WLOG can assume ergodicity.

② Show limit-amenable, PDR preserved by direct products w/ pmp
→ can assume ess. free & non-atomic by taking product with Bernoulli

③ THM 1 Limit-amenable ⇒ Poisson suspension is weakly Bernoulli
THM 2 Cost of Poisson suspension ≤ normalized cost.

⇒ By Abert-Weiss, max cost $\leq p$.

In particular, if $p = 1$, then Γ has fixed price 1.

Application : Metric groups

Goal:

Show for a countable group Γ w/ well-behaved metric d, there exists an infinite Γ-invt measure μ on the space of horofunctions $\mathcal{H}$ s.t. $\Gamma \curvearrowright (\mathcal{H}, \mu)$ is limit-amenable and doubly recurrent.

THM If d a left-invt, proper, approx. sub-additive, and satisfies the Overlapping neighborhoods property (ONP), then such an action exists.

Generalized word metrics

Defn Let d be an integer-valued quasi-metric on a countable group Γ. We say:

- d is proper if every ball of finite radius is finite
- d is left-invariant if $d(gh, gf) = d(h, f)$ $\forall f, g, h \in \Gamma$
- d is ε-approximately sub-additive if there $\exists$ an $\varepsilon > 0$ such that if

$$SS(\Gamma, n, \varepsilon) = \{x \in \Gamma : d(x, e) \in [n-\varepsilon, n+\varepsilon]\}$$

then $\forall\, n, m \geq \varepsilon$

$$SS(\Gamma, n, \varepsilon) \cdot SS(\Gamma, m, \varepsilon) \supset S(\Gamma, n+m).$$

proper + approx sub-additive $\Rightarrow$ growth rate exists

Overlapping neighborhoods property

(Γ, d) satisfies the ONP if $\exists\ C > 0$ s.t. $\forall\ m > 0$,

$$\boxed{\lim_{r\to\infty} \liminf_{n\to\infty} \frac{\#\{(x,y) \in B(n)^2 : |B(r) \cap B(x, n+C) \cap B(y, n+C)| < m\}}{|B(n)|^2} = 0}$$

Intuitively: if two radius n balls intersect non-trivially, then their radius-C nbhds are likely to have a large overlap which grows relative to n.

Horofunctions

Define $d_x(y) = d(x,y)$, $h_x = d_x - |x|$ so $h_x(e) = 0$.

Let $\mathcal{H}_0 = \overline{\{h_x : x \in \Gamma\}}$ in $\mathrm{Lip}_0(\Gamma) = \{h : \Gamma \to \mathbb{Z} : h$ is 1-Lipschitz $+\ h(e) = 0\}$.

Let $\mathcal{H} = \mathcal{H}_0 + \mathbb{Z} \subset \mathrm{Lip}(\Gamma)$ be the space of horofunctions on Γ.

$\mathcal{H}$ is locally compact because $\mathrm{Lip}_0(\Gamma)$ is compact.

$\Gamma \curvearrowright \mathcal{H}$ continuously by $g \cdot h(x) = h(g^{-1}x)$ for $g, x \in \Gamma$

Let $\partial\mathcal{H} = \mathcal{H}_0 \setminus \{h_x : x \in \Gamma\} + \mathbb{Z}$ be the horo boundary.

Measures on $\mathcal{H}$

Define Expand : $\mathrm{Lip}(\Gamma) \to \mathrm{Lip}(\Gamma)$ by $\mathrm{Expand}(\xi) = \xi - 1$

$\to$ Γ-equivariant, $\mathrm{Expand}(\mathcal{H}) = \mathcal{H}$.

Let $\mu_0 = \sum_{x \in \Gamma} \delta_{d_x}$, $\quad \mu_n = \dfrac{1}{|B(n)|} \sum_{x \in \Gamma} \delta_{d_x - n} = \dfrac{\mathrm{Expand}^n_* \mu_0}{|B(n)|}$

Then:

- μ_n is Γ-equivariant
- $\mu_n(\mathcal{H}_{\leq t}) = \dfrac{|B(n+t)|}{|B(n)|}$ where $\mathcal{H}_{\leq t} = \{ h \in \mathcal{H} : h(e) \leq t \}$
- non-amenability gives exponential decay, i.e. $\mu_n(\mathcal{H}_{\leq -m}(\Gamma)) \leq e^{-C \cdot m}$ for some constant C.

Spaces of Measures

Let $\mathrm{Meas}(\mathcal{H}) = \{\mu_n\}_{n=1}^{\infty}$, and $\overline{\mathrm{Meas}(\mathcal{H})}$ its closure in $\mathrm{Radon}(\mathcal{H})$ under the almost weak topology. That is:

$\nu_n \to \nu_\infty$ almost weakly $\iff \nu_n f \to \nu_\infty f \;\; \forall f : \mathcal{H} \to \mathbb{R}$ s.t. $\exists\, n \in \mathbb{N}$ w/ $\mathrm{supp}(f) = \mathcal{H}_{\leq n}$

Lemma For every $\mu \in \partial\mathrm{Meas}(\mathcal{H}) = \overline{\mathrm{Meas}(\mathcal{H})} \setminus \mathrm{Meas}(\mathcal{H})$, $\Gamma \curvearrowright (\mathcal{H}, \mu)$ is limit-amenable.

To get a PDR action, we prove:

Thm There exists a Γ-invt measure μ on $\mathcal{H}$ s.t. $\mu = \mathrm{Expand}_* \mu$ and $\mu = \int \nu \, d\xi(\nu)$ for $\xi \in \mathrm{Prob}(\partial\mathrm{Meas}(\mathcal{H}))$.

Double Recurrence

THM Let d a left-inv't, proper, approx. sub-additive, and satisfies the Overlapping neighborhoods property (ONP). Let μ be the measure satisfying the conclusion of the proposition. Then $\Gamma \curvearrowright (\mathcal{H}(\Gamma), \mu)$ is doubly recurrent.

1. Use ONP to show DR on a portion of X.

2. Use Expand map to extend everywhere.

Double recurrence + Limit amenability $\Rightarrow$ bdd cost.

Application

Thm For $i=1,2$, let d_i be a left-inv't proper integer valued quasi-metric on a countable group Γ_i, and let $\varepsilon>0$. Assume each (Γ_i, d_i) is ε-approx. sub additive, and d_1, d_2 have roughly comparable growth rates, i.e. for $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$ the ℓ^1 metric on $\Gamma_1 \times \Gamma_2$ the radius n ball $B(\Gamma, n)$, $B(\Gamma_i, n)$ in Γ, Γ_i satisfy

$$\lim_{n\to\infty} \frac{\# B(\Gamma_i, n)}{\# B(\Gamma, n)} = 0 \quad \text{for } i=1,2.$$

Then $\Gamma = \Gamma_1 \times \Gamma_2$ has fixed price 1.

$\boxed{\text{Sketch}}$

① Limit-amenable by theorem

② Show growth rate condition $\Rightarrow$ ONP

③ Need to show cost 1: take finite measure extension by $\mathrm{Cocycle}(\Gamma) = \{1\text{-Lipschitz cocycles } \Gamma \times \Gamma \to \mathbb{Z}^2\}$

$\to$ prove $\exists$ measure on $\mathrm{Cocycle} \times \Gamma$ which is still limit-amenable

$\to$ PDR is maintained

$\Rightarrow$ Action has cost 1.